

ON DOUBLE SHIFTED CONVOLUTION SUM OF $SL(2, \mathbb{Z})$ HECKE EIGEN FORMS

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ABSTRACT. Let $\lambda_i(n)$ $i = 1, 2, 3$ denote the normalised Fourier coefficients of holomorphic eigenform or Maass cusp form. In this paper we shall consider the sum:

$$S := \frac{1}{H} \sum_{h \leq H} V\left(\frac{h}{H}\right) \sum_{n \leq N} \lambda_1(n) \lambda_2(n+h) \lambda_3(n+2h) W\left(\frac{n}{N}\right),$$

where V and W are smooth bump functions, supported on $[1, 2]$. We shall prove a nontrivial upper bound, under the assumption that $H \geq N^{1/2+\epsilon}$.

1. INTRODUCTION

The study of the shifted convolution sum $D_k(N, h) := \sum_{N < n \leq 2N} d_k(n) d_k(n+h)$ of generalised divisor function $d_k(n)$ is a central problem in number theory, where $d_k(n)$ is defined to be the Dirichlet coefficient of $\zeta^k(s)$ in the half plane $\Re(s) > 1$. The sum $D_k(N, h)$ comes naturally in the computation of $2k$ -th power moment of Riemann Zeta function, which is defined as

$$I_k(T) := \int_1^T |\zeta(s)|^{2k} ds.$$

The general additive divisor problem consists of estimation of the quantity $\Delta_k(x, h)$, which is given by the equation

$$(1.1) \quad \sum_{n \leq x} d_k(n) d_k(n+h) = x P_{2k-2}(\log x; h) + \Delta_k(x, h),$$

where $k \geq 2$ is a fixed integer, $P_{2k-2}(\log x; h)$ is a suitable polynomial of degree $2k-2$ in $\log x$ whose coefficients depend on k and h , and $\Delta_k(x, h)$ is supposed to be the error term. From the above analogy, it is expected that $D_k(N, h)$ should be asymptotic to $c_{k,h} N \log^{2k-2} N$, for some constant $c_{k,h} > 0$, uniformly for h in some range. Even for a fixed h , this has only been proved for $k \leq 2$, and no proof exists for $k \geq 3$.

The behaviour of $D_k(N, h)$ for $k = 1$ is very simple. For $k = 2$, Ingham ([9]) first established the asymptotic formula that

$$D_2(N, h) \sim \frac{6}{\pi^2} \sigma_{-1}(h) N \log^2 N,$$

for any $h \in \mathbb{N}$, where $\sigma_{-1}(h) = \sum_{j|h} j^{-1}$. Later, T. Estermann established the asymptotic expansion (see [6])

$$\sum_{n \leq x} d(n) d(n+h) = x P_h(\log x) + O\left(x^{11/12} \log^x\right),$$

where $P_h(x)$ is a quadratic polynomial with leading coefficient $\frac{6}{\pi^2} \sigma_{-1}(h)$. Many authors have since revisited this problem. The best known results for the error term are due to Duke, Friedlander and Iwaniec [5] and Meurman [17].

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For $k = 3$, several authors have studied the property of $D_3(N, h)$. Some recent results are given in [1] and [10]. Averaging over shift $h \leq H$, Baier, Browning, Marasingha and Zhao ([1]) have given an asymptotic formula for $G(N, H) := \sum_{h \leq H} D(N, h)$, provided $N^{1/6+\epsilon} \leq H \leq N^{1-\epsilon}$. Precisely, they proved that

$$\sum_{h \leq H} \Delta(N, h) \ll (H^2 + H^{1/2} N^{13/12}) N^\epsilon,$$

where $1 \leq H \leq N$, and $\sum_{h \leq H} \Delta(N, h)$ is given by equation (1.1). For $k \geq 3$ the behaviour of $\Delta(N, h)$ has been studied by Ivić and Wu ([10]). They proved that, for $k \geq 3$ we have

$$\sum_{h \leq H} \Delta_k(N, h) \ll (H^2 + N^{1+\beta_k}) N^\epsilon \quad (1 \leq H \leq N),$$

where β_k is defined by

$$\beta_k := \inf \left\{ b_k : \int_1^X |\Delta(x)|^2 dx \ll X^{1+2b_k} \right\},$$

where $\Delta(x)$ is defined by equation

$$D_k(x) := \sum_{n \leq x} d_k(n) = xp_{k-1}(\log x) + \Delta(x).$$

In this paper we will study the behaviour of double shifted convolution sum of the coefficients of holomorphic cusp forms of weight k , or Maass eigenforms on the full modular group $SL(2, \mathbb{Z})$ (for details see for example [11] and [12]). We shall denote the space of such form by $\mathfrak{S}(\mathbb{Z})$. Let $f : \mathbb{H} \rightarrow \mathbb{C}$ be such that $f \in \mathfrak{S}(\mathbb{Z})$. Since $f(z+1) = f(z)$, f admits a Fourier expansion at infinity and we denote its normalised n th Fourier coefficient by $\lambda_f(n)$.

Double shifted convolution sum for Fourier coefficients is given by:

$$(1.2) \quad S(N, h) := \sum_{n \leq N} \lambda_1(n) \lambda_2(n+h) \lambda_3(n+2h),$$

where $\lambda_i \in \mathfrak{S}(\mathbb{Z})$ for $i = 1, 2, 3$. Summing over all shifts $h \leq H$ we shall prove the following theorem:

Theorem 1.1. *With $S(N, h)$ defined as above, assume that $H \gg N^{1/2+\epsilon}$, where $\delta > 0$. Then there exists a positive constant $\delta = \delta(\epsilon)$ such that*

$$\frac{1}{H} \sum_{h \leq H} V\left(\frac{h}{H}\right) \sum_{n \leq N} \lambda_1(n) \lambda_2(n+h) \lambda_3(n+2h) W\left(\frac{n}{N}\right) \ll N^{1-\delta},$$

where V and W are smooth bump functions, supported on the interval $[1, 2]$.

One of the motivation for studying the double shifted convolution sum for Fourier coefficients is the corresponding problem for the divisor function. Successful analysis of the sum

$$(1.3) \quad T_h(X) = \sum_{n \leq X} d(n-h) d(n) d(n+h)$$

has not been completed yet, even for a single positive integer h . It is conjectured that $T_h(X) \sim c_h X \log^3 X$, for a suitable constant $c_h > 0$. A heuristic analysis based on the underlying Diophantine equations suggests that one should take

$$c_h = \frac{11}{8} f(h) \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right),$$

where f is a suitable multiplicative function. It has been proved by Browning that (see [3, Theorem 2])

$$(1.4) \quad \sum_{h \leq H} (T_h(X) - c_h X \log^3 X) = o(HX \log^3 X),$$

provided $H \geq X^{3/4+\epsilon}$ ($\epsilon > 0$).

Recently, Valentine Blomer improved the range of H substantially to $X^{1/3+\epsilon}$ by using spectral theory of automorphic forms. His method is flexible enough to adopt for more general correlation sum. He proves the following theorem

Theorem: Blomer ([2]). *Let W be a smooth function with compact support in $[1, 2]$ with Mellin transform \widehat{W} . Let $1 \leq H \leq N/3$ and let $k \geq 2$ be an integer. Let a_n , $N \leq n \leq 2N$, be any sequence of complex numbers and let $r_d(n)$ denotes the Ramanujan sum. Then*

$$\begin{aligned} \sum_{h \leq H} W\left(\frac{h}{H}\right) \sum_{N \leq n \leq 2N} a(n) \tau(n-h) \tau(n+h) &= H \widehat{W}(1) \sum_{N \leq n \leq 2N} a(n) \sum_d \frac{r_d(2n)}{d^2} (\log n + 2\gamma - 2 \log d)^2 \\ &\quad + O\left(N^\epsilon \left(\frac{H^2}{N^{1/2}} + HN^{1/4} + (HN)^{1/2} + \frac{N}{H^{1/2}}\right) \|a\|_2\right), \end{aligned}$$

where the O -constant depends on (the Sobolev norms of) W and ϵ , and $\|a\|$ is ℓ^2 -norm.

Using this, Blomer obtains the following corollary

Corollary 1. *Let W be a smooth function with compact support in $[1, 2]$ with Mellin transform \widehat{W} . Let $1 \leq H \leq N/3$ and let $k \geq 2$ be an integer. Then*

$$(1.5) \quad \begin{aligned} \sum_{h \leq H} W\left(\frac{h}{H}\right) \sum_{N \leq n \leq 2N} \tau_k(n) \tau(n-h) \tau(n+h) &= \widehat{W}(1) H N Q_{k+1}(\log N) \\ &\quad + O\left(N^\epsilon \left(H^2 + NH^{1/2} + N^{3/2} H^{-1/2} + HN^{1-\frac{1}{k+2}}\right)\right), \end{aligned}$$

where τ_k denotes the k -fold divisor function, $Q_k + 1$ is a polynomial (depending only on k) of degree $k + 1$ and leading constant,

$$\frac{1}{(k-1)!} \prod_p \left(1 + \frac{1}{p}\right) \left(1 - \frac{1}{p^{\delta_p=2}} \left(1 - \frac{1}{p+1}\right)^k\right),$$

and the implied constant in the error term depends on (the Sobolev norms of) W , k and ϵ .

The above asymptotic formula is non trivial (in fact with a power saving error term) for

$$N^{1/3+\epsilon} \leq H \leq N^{1-\epsilon},$$

and it is independent of k . In the case of $k = 2$, this improves Browning's result substantially. Here, the lower bound for H is coming from the third error term of equation (1.5), which is the limit of the automorphic forms machinery. It is worth noting that the divisor function can be viewed as Fourier coefficients of Eisenstein series. Blomer remarked that using Jutila's circle method, one can prove analogous result for Fourier coefficients of cusp form in place of divisor function. Using this idea, Yongxiao Lin [15] proved the following theorem:

Theorem: Lin ([15]). *Let $1 \leq H \leq X/3$. Let W be a smooth function with compact support in $[1, 2]$, and a_n , $X \leq n \leq 2X$, be any sequence of complex numbers. Let $\lambda_1(n)$, $\lambda_2(n)$ be Hecke eigenvalues of holomorphic Hecke eigencuspforms of weight κ_1 , κ_2 for $SL(2, \mathbb{Z})$ respectively. Then*

$$\sum_{h \leq H} W\left(\frac{h}{H}\right) \sum_{N \leq n \leq 2N} a(n) \lambda_1(n-h) \lambda_2(n+h) \ll N^\epsilon \frac{N}{H} \left((HN)^{1/2} + \frac{N}{H^{1/2}}\right) \|a\|_2.$$

The error term coming here is comparable to the third and fourth expressions in O -term of Blomer's theorem, and both comes from the spectral theory of automorphic forms. As a consequence, Y. Lin proved the following corollary:

Corollary 2. Let $1 \leq H \leq X/3$. Let W be a smooth function with compact support in $[1, 2]$. Let $\lambda_1(n)$, $\lambda_2(n)$ and $\lambda_3(n)$ be Hecke eigenvalues of holomorphic Hecke eigencuspforms of weight κ_1 , κ_2 and κ_3 for $SL(2, \mathbb{Z})$ respectively. Then

$$\sum_{h \leq H} W\left(\frac{h}{H}\right) \sum_{N \leq n \leq 2N} \lambda_1(n-h) \lambda_2(n) \lambda_3(n+h) \ll N^\epsilon \min\left(NH, \frac{N^2}{H^{1/2}}\right).$$

Note that the result is non-trivial only for $H \geq N^{2/3+\epsilon}$, for any $\epsilon > 0$.

We use the circle method of Heath brown (see [8]) twice to improve the range of H and the same method also works in the case of Maass forms. By using the Voronoi summation formula for the eigenforms on congruence subgroups, given in appendix A.4 of [14], it is possible to prove this theorem for eigenform on the congruence subgroups of $SL(2, \mathbb{Z})$. The calculations will be similar.

2. Notations and Preliminaries

We shall first recall some basic facts about $SL(2, \mathbb{Z})$ automorphic forms. Our requirement is minimal. In fact, the Voronoi summation formula and cancellation in additive twist (see equation (2.2)) is all that we will be using. Let $f(z)$ ($z = x + iy, y > 0$) be a primitive holomorphic Hecke eigenform of integral weight $k(> 2)$ on the full modular group $SL(2, \mathbb{Z})$. The normalised Fourier expansion of f at cusp ∞ is given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz) \quad (\lambda_f(1) = 1),$$

where $e(z) = e^{2\pi iz}$ and $\lambda_f(1) = 1$. From Ramanujan-Petersson conjecture, which has been proved by Deligne we have $\lambda_f(n) \leq d(n)$ for every positive integer n . Analogously, let $f(z)$ be a primitive Maass cusp form on the group $SL(2, \mathbb{Z})$ with Laplacian eigenvalue $\frac{1}{4} + \nu^2$. Then the normalised Fourier expansion of f at cusp ∞ is given by

$$\sqrt{y} \sum_{n \neq 0} \lambda_f(n) K_{i\nu}(2\pi|n|y) e(nx),$$

where $K_{i\nu}$ denotes the K -Bessel function and $\lambda_f(1) = 1$. It follows from the Rankin-Selberg theory that the Fourier coefficients $\lambda_f(n)$'s are bounded on average, namely

$$(2.1) \quad \sum_{n \leq X} |\lambda_f(n)|^2 = C_f X + O\left(x^{3/5}\right),$$

for some constant $C_f > 0$. Ramanujan-Petersson conjecture predicts that $\lambda_f(n) \ll n^\epsilon$. This has been proved by Deligne in case of holomorphic cusp forms, where he proves that $\lambda_f(n) \leq d(n)$. In case of Maass cusp form the best known result is $\lambda_f(n) \ll n^{7/64+\epsilon}$, proved by Kim and Sarnak (see [13]). On the other hand, one knows that the Fourier coefficients oscillate quite substantially. For any $X > 1$ and any $\alpha \in \mathbb{R}$, we have

$$(2.2) \quad \sum_{n \leq X} \lambda_f(n) e(\alpha n) \ll_f X^{\frac{1}{2}} \log(2X), \quad \text{and} \quad \sum_{n \leq X} \lambda_f(n) \ll_{f, \epsilon} X^{\frac{1}{3}+\epsilon}.$$

where the implied constant depends only on f , and not on α (see for example [11, Page 71, Theorem 5.3] and [7]).

Notations : In our work $\tau(n)$ or $d(n)$ denotes the divisor function. ϵ, δ denote small positive constants, which may be different on different occurrence. All of the implied constant in this paper depends on these parameters.

3. Some Lemmas

Lemma 3.1. *We define*

$$(3.1) \quad \delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

For any integer $Q > 1$ there is a positive constant c_Q and a smooth function $h(x, y)$ defined on $(0, \infty) \times (-\infty, \infty)$ such that

$$(3.2) \quad \delta(n) = \frac{c_Q}{Q^2} \sum_{q=1}^{\infty} \sum'_{a \pmod{q}} e\left(\frac{an}{q}\right) h\left(\frac{q}{Q}, \frac{n}{Q^2}\right).$$

The constant c_Q satisfies

$$c_Q = 1 + O_N(Q^{-N}),$$

for any $N > 0$. Moreover $h(x, y) \ll x^{-1}$ for all y , and $h(x, y)$ is non-zero only when $x \leq \max\{1, 2|y|\}$. The smooth function $h(x, y)$ satisfies

$$(3.3) \quad x^i \frac{\partial h}{\partial x^i}(x, y) \ll_i x^{-1} \quad \text{and} \quad \frac{\partial h}{\partial y} = 0,$$

for $x \leq 1$ and $|y| \leq \frac{x}{2}$. And also for $|y| \geq \frac{x}{2}$, we have

$$(3.4) \quad x^i y^j \frac{\partial^{i+j} h(x, y)}{\partial x^i \partial y^j} \ll_{i,j} \frac{1}{x}.$$

Proof. See [8]. □

Lemma 3.2. *Let N, m and n be non-negative integers. Let $x = o(\min\{1, |y|\})$. Then we have*

$$\frac{\partial^{m+n} h(x, y)}{\partial x^m \partial y^n} \ll_{m,n,N} \frac{1}{x^{1+m+n}} \left(x^N + \min \left\{ 1, \left(\frac{x}{|y|} \right)^N \right\} \right).$$

The term x^N on the right may be omitted for $n = 0$.

Proof. See [8, Lemma 5] □

Next we recall the Poisson summation formula.

Lemma 3.3. Poisson summation formula: *$f : \mathbb{R} \rightarrow \mathbb{R}$ is any Schwarz class function. Fourier transform of f is defined as*

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x) e(-xy) dx,$$

where dx is the usual Lebesgue measure on \mathbb{R} . We have

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \widehat{f}(m).$$

If $W(x)$ is any smooth and compactly supported function on \mathbb{R} , we have:

$$\sum_{n \in \mathbb{Z}} e\left(\frac{an}{q}\right) W\left(\frac{n}{X}\right) = \frac{X}{q} \sum_{m \in \mathbb{Z}} \sum_{\alpha \pmod{q}} e\left(\frac{\alpha + m}{q}\right) \widehat{W}\left(\frac{mX}{q}\right).$$

Proof. See [12, page 69]. □

Remark: 1. If $W(x)$ satisfies $x^j W^j(x) \ll 1$, then it can be easily shown, by integrating by parts that dual sum is essentially supported on $m \ll \frac{q(qX)^\epsilon}{X}$. The contribution coming from $m \gg \frac{q(qX)^\epsilon}{X}$ is negligibly small.

We shall recall the Voronoi summation formula for $SL(2, \mathbb{Z})$ automorphic forms. For sake of exposition we only present the case of Maass forms. The case of holomorphic forms is even simpler. Let f be a Maass form with Laplacian eigenvalue $1/4 + \nu^2$ and with Fourier expansion

$$\sqrt{y} \sum_{n \neq 0} \lambda(n) K_{i\nu}(2\pi|n|y) e(nx).$$

We shall use the following Voronoi type summation formula, which was first proved by Meruman [16].

Lemma 3.4. Voronoi summation formula: *Let h be a compactly supported smooth function on the interval $(0, \infty)$. We have*

$$(3.5) \quad \sum_{n=1}^{\infty} \lambda(n) e_q(an) h(n) = \frac{1}{q} \sum_{\pm} \sum_{n=1}^{\infty} \lambda(\mp n) e_q(\pm \bar{a}n) H^{\pm} \left(\frac{n}{q^2} \right),$$

where $a\bar{a} \equiv 1 \pmod{q}$, and

$$H^{-}(y) = \frac{-\pi}{\cosh(\pi\nu)} \int_0^{\infty} h(x) \{Y_{2i\nu} + Y_{-2i\nu}\} (4\pi\sqrt{xy}) dx,$$

$$H^{+}(y) = 4 \cosh(\pi\nu) \int_0^{\infty} h(x) K_{2i\nu} (4\pi\sqrt{xy}) dx,$$

where $Y_{2i\nu}$ and $K_{2i\nu}$ are Bessel's functions of first and second kind and $e_q(x) = e^{\frac{2\pi i x}{q}}$.

Remark: 2. When h is supported on the interval $[X, 2X]$ and satisfies $x^j h^{(j)}(x) \ll 1$, then integrating by parts and using the properties of Bessel's function it is easy to see that the sums on the right hand side of equation (3.5) are essentially supported on $n \ll_{f, \epsilon} q^2 (qX)^{\epsilon} / X$. For smaller values of n we will use trivial bound that is $H^{\pm} \left(\frac{n}{q^2} \right) \ll X$.

4. PROOF OF THEOREM 1.1

We shall prove the case when all f_1, f_2 and f_3 are Maass forms. The case of holomorphic eigenforms are similar (even relatively simple). We first substitute $n + h = m (\sim N)$ and $\delta(m, n) := \delta(m - n)$ where $\delta(n)$ is defined by equation (3.1). We have

$$S := \frac{1}{H} \sum_{h \leq H} \sum_{m, n \in \mathbb{Z}} \lambda_1(n) \lambda_2(m) \lambda_3(n + 2h) \delta(m, n) W_1 \left(\frac{n}{N} \right) W_2 \left(\frac{m}{N} \right) V \left(\frac{h}{H} \right),$$

where $W_1(x)$, $W_2(x)$ and $V(x)$ are smooth bump functions supported on the interval $[1, 2]$ and satisfying

$$x^{(j)} V^{(j)}(x), \quad x^{(j)} W_{\ell}^{(j)}(x) \ll 1, \quad \text{for } \ell = 1, 2 \quad \text{and } j \in \mathbb{Z}, j \geq 0.$$

By Lemma 3.1 we write

$$S = \frac{c_{Q_1}}{H Q_1^2} \sum_{h \leq H} \sum_{m, n \in \mathbb{Z}} \sum_{q_1 \leq Q_1} \sum'_{a_1(q_1)} \lambda_1(n) \lambda_2(m) \lambda_3(n + 2h) e \left(\frac{a_1(n + h - m)}{q_1} \right) \\ \times h \left(\frac{q_1}{Q_1}, \frac{n + h - m}{Q_1^2} \right) W_1 \left(\frac{n}{N} \right) W_2 \left(\frac{m}{N} \right) V \left(\frac{h}{H} \right).$$

We substitute $n + 2h = l (\sim N)$. We apply Lemma 3.1 once again to obtain

$$\begin{aligned}
S &= \frac{c_{Q_1} c_{Q_2}}{H Q_1^2 Q_2^2} \sum_{h \leq H} \sum_{m, n, l \in \mathbb{Z}} \sum_{q_1 \leq Q_1} \sum'_{a_1(q_1)} \sum_{q_2 \leq Q_2} \sum'_{a_2(q_2)} \lambda_1(n) \lambda_2(m) \lambda_3(l) e \left(\frac{a_1(n + h - m)}{q_1} \right) \\
&\quad \times \left(\frac{a_2(n + 2h - l)}{q_2} \right) h \left(\frac{q_1}{Q_1}, \frac{n + h - m}{Q_1^2} \right) h \left(\frac{q_2}{Q_2}, \frac{n + 2h - l}{Q_2^2} \right) W_1 \left(\frac{n}{N} \right) \\
&\quad \times W_2 \left(\frac{m}{N} \right) W_3 \left(\frac{l}{N} \right) V \left(\frac{h}{H} \right) \\
&= \frac{c_{Q_1} c_{Q_2}}{H Q_1^2 Q_2^2} \sum_{q_1 \leq Q_1} \sum'_{a_1(q_1)} \sum_{q_2 \leq Q_2} \sum'_{a_2(q_2)} \left(\sum_{n \in \mathbb{Z}} \lambda_1(n) e \left(\frac{a_1 q_2 + a_2 q_1}{q_1 q_2} n \right) W_1 \left(\frac{n}{N} \right) \right) \times \\
&\quad \left(\sum_{m \in \mathbb{Z}} \lambda_2(m) e \left(-\frac{a_1 m}{q_1} \right) W_2 \left(\frac{m}{N} \right) \right) \left(\sum_{l \in \mathbb{Z}} \lambda_3(l) e \left(-\frac{a_2 l}{q_2} \right) W_3 \left(\frac{l}{N} \right) \right) \\
&\quad \times \sum_{h \leq H} e \left(\frac{a_1 q_2 + 2a_2 q_1}{q_1 q_2} h \right) h \left(\frac{q_1}{Q_1}, \frac{n + h - m}{Q_1^2} \right) h \left(\frac{q_2}{Q_2}, \frac{n + 2h - l}{Q_2^2} \right) V \left(\frac{h}{H} \right).
\end{aligned}$$

4.1. Applying Poisson summation formula. We first write $h = \alpha + b q_1 q_2$, and then apply the Poisson summation formula in variable b . We set $Q_1 = Q_2 = \sqrt{X}$. We have

$$\begin{aligned}
&\sum_{h \in \mathbb{Z}} e \left(\frac{a_1 q_2 + 2a_2 q_1}{q_1 q_2} h \right) \times h \left(\frac{q_1}{Q_1}, \frac{n + h - m}{Q_1^2} \right) h \left(\frac{q_2}{Q_2}, \frac{n + 2h - l}{Q_2^2} \right) V \left(\frac{h}{H} \right) \\
&= \sum_{\alpha \pmod{q}} e \left(\frac{a_1 q_2 + 2a_2 q_1}{q_1 q_2} \alpha \right) \sum_{h \in \mathbb{Z}} \int_{\mathbb{R}} h \left(\frac{q_1}{Q_1}, \frac{n + \alpha + x q_1 q_2 - u}{Q_1^2} \right) \\
&\quad \times h \left(\frac{q_2}{Q_2}, \frac{n + 2(\alpha + x q_1 q_2) - v}{Q_2^2} \right) V \left(\frac{\alpha + x q_1 q_2}{H} \right) e(-hx) dx \\
(4.1) \quad &= \frac{H}{q_1 q_2} \sum_{h \in \mathbb{Z}} \mathfrak{C}(a_1, a_2, q_1, q_2) \mathfrak{J}(h; n, m, l, q_1, q_2),
\end{aligned}$$

after substituting $\frac{\alpha + x q_1 q_2}{H} = y$, where the character sum $\mathfrak{C}(a_1, a_2, q_1, q_2)$ is given by

$$(4.2) \quad \mathfrak{C}(a_1, a_2, q_1, q_2) = \sum_{\alpha \pmod{q}} e \left(\frac{a_1 q_2 + 2a_2 q_1}{q_1 q_2} \alpha \right)$$

and $\mathfrak{J}(m, n, r, q)$ is given by

$$\begin{aligned}
\mathfrak{J}(h; n, u, v, q_1, q_2) &= \int_{\mathbb{R}} h \left(\frac{q_1}{Q_1}, \frac{n + xH - u}{Q_1^2} \right) h \left(\frac{q_2}{Q_2}, \frac{n + 2xH - v}{Q_2^2} \right) \\
&\quad \times V(x) e \left(-\frac{hHx}{q_1 q_2} \right) dx.
\end{aligned}$$

Note that the x -integral is supported only on the interval $[1, 2]$. Applying integration-by-parts j times and bounds of function $V(x)$ and $h(x, y)$ (listed in Lemma 3.1) we have

$$\begin{aligned}
\mathfrak{J}(h; n, u, v, q_1, q_2) &= \int_{\mathbb{R}} \sum_{p_1+p_2+p_3=j} h^{(p_1)} \left(\frac{q_1}{Q_1}, \frac{n+xH-u}{Q_1^2} \right) h^{(p_2)} \left(\frac{q_2}{Q_2}, \frac{n+2xH-v}{Q_2^2} \right) \\
&\quad \times V^{(p_3)}(x) e \left(-\frac{hHx}{q_1q_2} \right) \left(\frac{q_1q_2}{hH} \right)^j dx \\
&\ll \left(\frac{q_1q_2}{hH} \right)^j \left\{ \sum_{p_1+p_2+p_3=j} \left(1 + \left(\frac{H}{Q_1^2} \right)^{p_1} \left(\frac{Q_1}{q_1} \right)^{p_1+1} + \left(\frac{H}{Q_2^2} \right)^{p_2} \left(\frac{Q_2}{q_2} \right)^{p_2+1} \right) \right\} \\
&\ll Q_1Q_2 \left(\frac{q_1q_2}{hH} + \frac{q_2}{hQ_1} + \frac{q_1}{hQ_2} \right)^j \ll Q_1Q_2 \left(\frac{q_1q_2}{hH} + \frac{1}{h} \right)^j.
\end{aligned}$$

We note that if $h \gg (QX)^\epsilon \left(\frac{q_1q_2}{hH} + 1 \right)$, then contribution of $\mathfrak{J}(h; n, m, l, q_1, q_2)$ is negligibly small i.e., of order $O_A(X^{-A})$ for any $A > 0$. Evaluating the character sum given in equation (4.2), we can write the right hand side of equation (4.1) as

$$(4.3) \quad = H \sum_{\substack{|h| \leq \frac{q_1q_2}{H} + 1 \\ a_1q_2 + 2a_2q_1 + h \equiv 0 \pmod{q_1q_2}}} \mathfrak{J}(h; n, m, l, q_1, q_2).$$

4.2. Applying Voronoi summation formula. We shall apply Voronoi summation formula simultaneously to sum over l and m , where compactly supported function $h(x)$ is replaced by $W_2(u)W_3(v)\mathfrak{J}(h; n, u, v, q_1, q_2)$. We have

$$\begin{aligned}
&\sum_{m, l \in \mathbb{Z}} \lambda_2(m) \lambda_3(l) e \left(-\frac{a_1m}{q_1} \right) e \left(-\frac{a_2l}{q_2} \right) W_2 \left(\frac{m}{N} \right) W_3 \left(\frac{l}{N} \right) \mathfrak{J}(h; n, m, l, q_1, q_2) \\
&= \frac{1}{q_1q_2} \sum_{m, l \ll \frac{Q^2(QX)^\epsilon}{N}} \lambda_2(m) \lambda_3(l) e \left(\frac{\overline{a_1}m}{q_1} \right) e \left(\frac{\overline{a_2}l}{q_2} \right) \mathcal{H}_h^\pm \left(n; \frac{m}{q_1}, \frac{l}{q_2} \right) + O_A(N^{-A}),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{H}_h^+(w; y, z) &= 4 \cosh(\pi\nu) \iint_{\mathbb{R}^2} W_2 \left(\frac{u}{N} \right) W_3 \left(\frac{v}{N} \right) \int_{\mathbb{R}} h \left(\frac{q_1}{Q_1}, \frac{w+xH-u}{Q_1^2} \right) h \left(\frac{q_2}{Q_2}, \frac{w+2xH-v}{Q_2^2} \right) \\
&\quad \times V(x) e \left(-\frac{hHx}{q_1q_2} \right) dx K_{2i\nu}(4\pi\sqrt{yu}) K_{2i\nu}(4\pi\sqrt{zv}) du dv.
\end{aligned}$$

we have similar expression for $\mathcal{H}_h^-(w; y, z)$, where Bessel function $K_{2i\nu}(x)$ is replaced by Bessel function $\{Y_{2i\nu} + Y_{-2i\nu}\}(x)$.

We first make change of variables $\frac{u}{N} = u'$ and $\frac{v}{N} = v'$, we have

$$\begin{aligned}
\mathcal{H}_h^+(w; y, z) &= N^2 4 \cosh(\pi\nu) \iint_{1 \leq u', v' \leq 2} W_2(u') W_3(v') \int_{\mathbb{R}} h \left(\frac{q_1}{Q_1}, \frac{w+xH-Nu'}{Q_1^2} \right) h \left(\frac{q_2}{Q_2}, \frac{w+2xH-Nv'}{Q_2^2} \right) \\
&\quad \times V(x) e \left(-\frac{hHx}{q_1q_2} \right) dx K_{2i\nu}(4\pi\sqrt{yNu'}) K_{2i\nu}(4\pi\sqrt{zNv'}) du' dv'.
\end{aligned}$$

We have used the asymptotic formula for $K_{2i\nu}(x)$, $Y_{2i\nu}(x)$ and $Y_{-2i\nu}(x)$ to show that dual sum over m and l is supported on $(QN)^\epsilon$, namely

$$(4.4) \quad J_{k-1}(x), Y_{\pm 2i\nu}(x) = e^{ix} U_{\pm 2i\nu}(x) + e^{-ix} \overline{U}_{\pm 2i\nu}(x) \quad \text{and} \quad \left| x^k K_\nu^{(k)}(x) \right| \ll_{k, \nu} \frac{e^{-x}(1 + \log|x|)}{(1+x)^{1/2}},$$

where the function $U_{\pm 2i\nu}(x)$ satisfies,

$$x^j U_{\pm 2i\nu}^{(j)}(x) \ll_{j, \nu, k} (1+x)^{-1/2}.$$

Since K Bessel function has exponential decay, hence integral is negligibly small for $\frac{m}{Q^2}N \gg (QN)^\epsilon$. In case of Y Bessel function, integrating by parts we have

$$\begin{aligned} \int_{\mathbb{R}} U_1(x) Y_{\pm 2i\nu} \left(4\pi \frac{\sqrt{nN}x}{q} \right) dx &= 2 \int_{\mathbb{R}} U_1(y^2) y Y_{\pm 2i\nu} \left(4\pi \frac{\sqrt{nN}}{q} y \right) dy \\ &= \int_{\mathbb{R}} U_1(y^2) y U_{\pm 2i\nu} \left(2 \frac{\sqrt{nN}}{q} y \right) e \left(2 \frac{\sqrt{nN}}{q} y \right) dy \\ &\ll M \left(\frac{Dq}{\sqrt{nN}} \right)^j, \end{aligned}$$

where $U_1(x)$ is smooth bump function supported on the interval $[N, 2N]$ and satisfies $x^j U_1^{(j)} \ll D^j$. Hence integral is negligibly small if $\frac{Dq}{\sqrt{nN}} \ll 1$, that is $n \gg \frac{q^2 D^2}{N}$. In our case, from Lemma 3.2 we have $D = Q/q$. So after Voronoi summation formula, summation over m and l are supported on $m, l \ll \frac{Q^2(QN)^\epsilon}{N}$.

We want to calculate the derivative of $\mathcal{H}_h^+(w; y, z)$ with respect to variable w . We first substitute the following change of variables, $w + xH - Nu' = Nu$ and $w + 2xH - Nv' = Nv'$. We have

$$\begin{aligned} \mathcal{H}_h^+(w; y, z) &= N^2 4 \cosh(\pi\nu) \iint_{\mathbb{R}^2} W_2 \left(\frac{w + xH - Nu}{N} \right) W_3 \left(\frac{w + 2xH - Nv}{N} \right) \int_{\mathbb{R}} h \left(\frac{q_1}{Q_1}, \frac{Nu}{Q_1^2} \right) \\ &\quad \times h \left(\frac{q_2}{Q_2}, \frac{Nv}{Q_2^2} \right) V(x) e \left(-\frac{hHx}{q_1 q_2} \right) dx K_{2i\nu} \left(4\pi \sqrt{y(w + xH - Nu)} \right) \\ &\quad \times K_{2i\nu} \left(4\pi \sqrt{z(w + 2xH - Nv)} \right) du dv. \end{aligned}$$

For $z > 0$, for all ν and $k \geq 0$, we have following bound for Bessels functions (see [14, Lemma C.1 and C.2])

$$(z^\nu J_\nu(z))' = z^\nu J_{\nu-1}(z), \quad (z^\nu K_\nu(z))' = z^\nu K_{\nu-1}(z), \quad (z^\nu Y_\nu(z))' = z^\nu Y_{\nu-1}(z),$$

Using Lemma 3.2 and above bounds for the Bessels functions we have:

$$\begin{aligned}
& \frac{\partial}{\partial w} \mathcal{H}_h^+(w; y, z) \\
&= \frac{\partial}{\partial w} N^2 \frac{4 \cosh(\pi \nu)}{Q_1^2} \iint_{u \ll \frac{q_1}{Q_1}, v \ll \frac{q_2}{Q_2}} W_2 \left(\frac{w + xH - Nu}{N} \right) \\
&\quad \times W_3 \left(\frac{w + 2xH - Nv}{N} \right) \int_{\mathbb{R}} h \left(\frac{q_1}{Q_1}, \frac{Nu}{Q_1^2} \right) h \left(\frac{q_2}{Q_2}, \frac{Nu}{Q_1^2} \right) V(x) e \left(-\frac{hHx}{q_1 q_2} \right) \\
&\quad \times K_{2i\nu} \left(4\pi \sqrt{y(w + xH - Nu)} \right) K_{2i\nu} \left(4\pi \sqrt{z(w + 2xH - Nv)} \right) du dv dx \\
&\ll N^2 \frac{Q_1}{q_1} \frac{Q_2}{q_2} \iint_{u \ll \frac{q_1}{Q_1}, v \ll \frac{q_2}{Q_2}} \left(\frac{1}{N} + \frac{1}{N} \right. \\
&\quad \left. + \frac{\sqrt{y}}{\sqrt{w + xH - Nu}} K'_{2i\nu} \left(4\pi \sqrt{y(w + xH - Nu)} \right) + \dots \right) du dv dx \\
&\ll N^2 \frac{Q_1}{q_1} \frac{Q_2}{q_2} \iint_{u \ll \frac{q_1}{Q_1}, v \ll \frac{q_2}{Q_2}} \left(\frac{1}{N} + \frac{1}{N} \right. \\
&\quad \left. + \frac{1}{N} \sqrt{y(w + xH - Nu)} K'_{2i\nu} \left(4\pi \sqrt{y(w + xH - Nu)} \right) + \dots \right) du dv dx \\
(4.5) \quad &\ll N^2 \frac{Q_1}{q_1} \frac{Q_2}{q_2} \frac{1}{N} \frac{q_1}{Q_1} \frac{q_2}{Q_2} \ll N,
\end{aligned}$$

as x -integral is supported on the interval $[1, 2]$.

From the congruence relation given in the equation (4.3), we have

$$a_1 q_2 + 2a_2 q_1 + h \equiv 0(q_1 q_2) \Rightarrow a_1 q_2 + h \equiv 0(q_1) \quad \text{and} \quad 2a_2 q_1 + h \equiv 0(q_2).$$

After the application of Poisson and Voronoi summation formulae, we have:

$$\begin{aligned}
S &= \frac{c_{Q_1} c_{Q_2}}{Q_1^2 Q_2^2} \sum_{q_1 \leq Q_1} \sum_{q_2 \leq Q_2} \sum_{m, l \ll X^\epsilon} \sum_{h \leq \left(\frac{q_1 q_2}{H} + 1\right)} \frac{1}{q_1 q_2} \lambda_2(m) \lambda_3(l) e \left(\frac{\overline{a_1} m}{q_1} \right) e \left(\frac{\overline{a_2} l}{q_2} \right) \times \\
(4.6) \quad &\sum_{n \in \mathbb{Z}} \lambda_1(n) e \left(\frac{-h - a_2 q_1}{q_1 q_2} n \right) W_1 \left(\frac{n}{N} \right) \mathcal{H}_h^\pm \left(n; \frac{m}{q_1^2}, \frac{l}{q_2^2} \right)
\end{aligned}$$

Now we use Riemann Stieltjes integral to evaluate last sum in above equation. For any $\alpha \in \mathbb{R}$, by using cancellation in additive twist (see equations (2.2)) and equation (4.5), we have

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} \lambda_1(n) e(n\alpha) W_1 \left(\frac{n}{N} \right) \mathcal{H}_h^\pm \left(n; \frac{m}{q_1^2}, \frac{l}{q_2^2} \right) &= \int_N^{2N} \left(\sum_{n \leq y} \lambda_1(n) e(n\alpha) \right) \frac{\partial}{\partial y} \mathcal{H}_h^\pm \left(y; \frac{m}{q_1^2}, \frac{l}{q_2^2} \right) dy \\
(4.7) \quad &\ll \int_N^{2N} y^{1/2} N dy \ll N^{5/2}.
\end{aligned}$$

Substituting bound of equation (4.7) in equation (4.6) we have

$$\begin{aligned}
S &\ll \frac{c_{Q_1} c_{Q_2}}{Q_1^2 Q_2^2} \sum_{q_1 \leq Q_1} \sum_{q_2 \leq Q_2} \sum_{m, l \ll X^\epsilon} \sum_{h \leq \left(\frac{q_1 q_2}{H} + 1\right)} \frac{1}{q_1 q_2} N^{5/2} \\
&\ll \frac{c_{Q_1} c_{Q_2} N^{5/2+\epsilon}}{Q_1^2 Q_2^2} \sum_{q_1 \leq Q_1} \sum_{q_2 \leq Q_2} \frac{1}{q_1 q_2} \left(\frac{q_1 q_2}{H} + 1 \right) \ll \frac{N^{3/2+\epsilon}}{H} + N^{1/2+\epsilon} \ll \frac{N^{3/2+\epsilon}}{H}.
\end{aligned}$$

as Q_1 and $Q_2 = \sqrt{N}$.

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